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Research on Two Fractional Integrals of Fractional Trigonometric Functions

Chii-Huei Yu

School of Mathematics and Statistics,

Zhaoqing University, Guangdong, China

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***Abstract*: In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we study two fractional integral problems. We can obtain the exact solutions of these two fractional integrals by using some techniques. In addition, our results are generalizations of traditional calculus results.**

***Keywords*: Jumarie type of R-L fractional calculus, new multiplication, fractional analytic functions, exact solutions, fractional integrals.**

I. INTRODUCTION

Fractional calculus with derivatives and integrals of any real or complex order has its origin in the work of Euler, and even earlier in the work of Leibniz. Shortly after being introduced, the new theory turned out to be very attractive to many famous mathematicians and scientists, for example, Laplace, Riemann, Liouville, Abel, and Fourier. Fractional calculus has important applications in many scientific fields such as physics, mechanics, biology, electrical engineering, viscoelasticity, control theory, economics, and so on [1-13].

However, the definition of fractional derivative is not unique. Commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, Jumarie's modified R-L fractional derivative [14-18]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we study the following two α-fractional integrals:

$$\left({}_0I_x^{\alpha}\right)\left[\left[[cos_{\alpha}(x^{\alpha})]^{\otimes_{\alpha}4}+[sin_{\alpha}(x^{\alpha})]^{\otimes_{\alpha}4}\right]^{\otimes_{\alpha}(-1)}\right],$$

and

$$\left({}_0I_x^{\alpha}\right)\left[\left[[cos_{\alpha}(x^{\alpha})]^{\otimes_{\alpha}6}+[sin_{\alpha}(x^{\alpha})]^{\otimes_{\alpha}6}\right]^{\otimes_{\alpha}(-1)}\right].$$

Where $0<\alpha\leq 1$. Using some techniques, the exact solutions of these two fractional integrals can be obtained. In fact, our results are generalizations of the results of classical calculus.

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II. PRELIMINARIES

At first, we introduce the fractional calculus used in this paper.

Definition 2.1 ([19])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{1}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{2}$$

where $\Gamma(\ )$ is the gamma function.

In the following, some properties of Jumarie type of R-L fractional derivative are introduced.

Proposition 2.2 ([20])**:** *If* α,β,x_0,c *are real numbers and* $\beta\geq\alpha>0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[c]=0. \tag{4}$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([21])**:** If x,x_0, and a_k are real numbers for all k, $x_0\in(a,b)$, and $0<\alpha\leq 1$. If the function $f_{\alpha}:[a,b]\to R$ can be expressed as an α-fractional power series, i.e., $f_{\alpha}(x^{\alpha})=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_{\alpha}(x^{\alpha})$ is α-fractional analytic at x_0. Furthermore, if $f_{\alpha}:[a,b]\to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_{α} is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([22])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. If $f_{\alpha}(x^{\alpha})$ and $g_{\alpha}(x^{\alpha})$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_{\alpha}(x^{\alpha})=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{5}$$

$$g_{\alpha}(x^{\alpha})=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\ . \tag{6}$$

Then we define

$$\begin{aligned}&f_{\alpha}(x^{\alpha})\otimes_{\alpha}g_{\alpha}(x^{\alpha})\\&=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\otimes_{\alpha}\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\\&=\sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}.\end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned}&f_{\alpha}(x^{\alpha})\otimes_{\alpha}g_{\alpha}(x^{\alpha})\\&=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_{\alpha}n}\otimes_{\alpha}\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_{\alpha}n}\\&=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_{\alpha}n}.\end{aligned} \tag{8}$$

Definition 2.5 ([23])**:** If $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{a_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha\right)^{\otimes_\alpha n}, \tag{9}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{b_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha\right)^{\otimes_\alpha n}. \tag{10}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha\big(g_\alpha(x^\alpha)\big) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \big(g_\alpha(x^\alpha)\big)^{\otimes_\alpha n}, \tag{11}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha\big(f_\alpha(x^\alpha)\big) = \sum_{n=0}^{\infty} \frac{b_n}{n!} \big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n}. \tag{12}$$

Definition 2.6 ([24])**:** Let $0 < \alpha \leq 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha \circ g_\alpha)(x^\alpha) = (g_\alpha \circ f_\alpha)(x^\alpha) = \frac{1}{\Gamma(\alpha+1)} x^\alpha. \tag{13}$$

Then $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are called inverse functions of each other.

Definition 2.7 ([25])**:** If $0 < \alpha \leq 1$, and x is a real variable. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes_\alpha n}. \tag{14}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$. On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^k x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes_\alpha 2n}, \tag{15}$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{16}$$

Definition 2.8 ([26])**:** Let $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n} = f_\alpha(x^\alpha) \otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha (-1)}$.

Definition 2.9 ([27])**:** The smallest positive real number T_α such that $E_\alpha(iT_\alpha) = 1$, is called the period of $E_\alpha(ix^\alpha)$.

III. MAIN RESULTS

In this section, we solve two fractional integrals of fractional trigonometric functions.

Theorem 3.1: *If* $0 < \alpha \leq 1$, *then*

$$\left({}_0I_x^\alpha\right)\left[\left[[cos_\alpha(x^\alpha)]^{\otimes_\alpha 4} + [sin_\alpha(x^\alpha)]^{\otimes_\alpha 4}\right]^{\otimes_\alpha (-1)}\right] = \frac{1}{\sqrt{2}} \cdot arctan_\alpha\left(\frac{1}{\sqrt{2}} tan_\alpha(2x^\alpha)\right). \tag{17}$$

Proof $\left({}_0I_x^\alpha\right)\left[\left[[cos_\alpha(x^\alpha)]^{\otimes_\alpha 4} + [sin_\alpha(x^\alpha)]^{\otimes_\alpha 4}\right]^{\otimes_\alpha (-1)}\right]$

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$$= \left({}_{0}I_{x}^{\alpha} \right) \left[\left[\left([cos_\alpha(x^\alpha)]^{\otimes_\alpha 2} + [sin_\alpha(x^\alpha)]^{\otimes_\alpha 2} \right)^{\otimes_\alpha 2} - 2[cos_\alpha(x^\alpha)]^{\otimes_\alpha 2} \otimes_\alpha [sin_\alpha(x^\alpha)]^{\otimes_\alpha 2} \right]^{\otimes_\alpha (-1)} \right]$$

$$= \left({}_{0}I_{x}^{\alpha} \right) \left[\left[1 - 2[cos_\alpha(x^\alpha)]^{\otimes_\alpha 2} \otimes_\alpha [sin_\alpha(x^\alpha)]^{\otimes_\alpha 2} \right]^{\otimes_\alpha (-1)} \right]$$

$$= \left({}_{0}I_{x}^{\alpha} \right) \left[\left[1 - \frac{1}{2}[sin_\alpha(2x^\alpha)]^{\otimes_\alpha 2} \right]^{\otimes_\alpha (-1)} \right]$$

$$= \left({}_{0}I_{x}^{\alpha} \right) \left[2\left[2 - [sin_\alpha(2x^\alpha)]^{\otimes_\alpha 2} \right]^{\otimes_\alpha (-1)} \right]$$

$$= \left({}_{0}I_{x}^{\alpha} \right) \left[\left[2 - [sin_\alpha(2x^\alpha)]^{\otimes_\alpha 2} \right]^{\otimes_\alpha (-1)} \otimes_\alpha \left({}_{0}D_{x}^{\alpha} \right)[2x^\alpha] \right]$$

$$= \left({}_{0}I_{x}^{\alpha} \right) \left[[sec_\alpha(2x^\alpha)]^{\otimes_\alpha 2} \otimes_\alpha \left[2[sec_\alpha(2x^\alpha)]^{\otimes_\alpha 2} - [tan_\alpha(2x^\alpha)]^{\otimes_\alpha 2} \right]^{\otimes_\alpha (-1)} \otimes_\alpha \left({}_{0}D_{x}^{\alpha} \right)[2x^\alpha] \right]$$

$$= \left({}_{0}I_{x}^{\alpha} \right) \left[\left[2[sec_\alpha(2x^\alpha)]^{\otimes_\alpha 2} - [tan_\alpha(2x^\alpha)]^{\otimes_\alpha 2} \right]^{\otimes_\alpha (-1)} \otimes_\alpha \left({}_{0}D_{x}^{\alpha} \right)[tan_\alpha(2x^\alpha)] \right]$$

$$= \left({}_{0}I_{x}^{\alpha} \right) \left[\left[2\left[1 + [tan_\alpha(2x^\alpha)]^{\otimes_\alpha 2} \right] - [tan_\alpha(2x^\alpha)]^{\otimes_\alpha 2} \right]^{\otimes_\alpha (-1)} \otimes_\alpha \left({}_{0}D_{x}^{\alpha} \right)[tan_\alpha(2x^\alpha)] \right]$$

$$= \left({}_{0}I_{x}^{\alpha} \right) \left[\left[\left[2 + [tan_\alpha(2x^\alpha)]^{\otimes_\alpha 2} \right] \right]^{\otimes_\alpha (-1)} \otimes_\alpha \left({}_{0}D_{x}^{\alpha} \right)[tan_\alpha(2x^\alpha)] \right]$$

$$= \frac{1}{\sqrt{2}} \cdot arctan_\alpha \left(\frac{1}{\sqrt{2}} tan_\alpha(2x^\alpha) \right).$$

Q.e.d.

Theorem 3.2: *Let* $0 < \alpha \leq 1$, *and* $x \neq 0$, *then*

$$\left({}_{0}I_{x}^{\alpha} \right) \left[\left[[cos_\alpha(x^\alpha)]^{\otimes_\alpha 6} + [sin_\alpha(x^\alpha)]^{\otimes_\alpha 6} \right]^{\otimes_\alpha (-1)} \right] = arctan_\alpha\left(tan_\alpha(x^\alpha) - cot_\alpha(x^\alpha) \right) + \begin{cases} \frac{T_\alpha}{4} & if\ x > 0 \\ -\frac{T_\alpha}{4} & if\ x < 0 \end{cases}. \tag{18}$$

Proof $\left({}_{0}I_{x}^{\alpha} \right) \left[\left[[cos_\alpha(x^\alpha)]^{\otimes_\alpha 6} + [sin_\alpha(x^\alpha)]^{\otimes_\alpha 6} \right]^{\otimes_\alpha (-1)} \right]$

$$= \left({}_{0}I_{x}^{\alpha} \right) \left[\left[[tan_\alpha(x^\alpha)]^{\otimes_\alpha 6} + 1 \right]^{\otimes_\alpha (-1)} \otimes_\alpha [sec_\alpha(x^\alpha)]^{\otimes_\alpha 6} \right]$$

$$= \left({}_{0}I_{x}^{\alpha} \right) \left[\left[[tan_\alpha(x^\alpha)]^{\otimes_\alpha 6} + 1 \right]^{\otimes_\alpha (-1)} \otimes_\alpha [sec_\alpha(x^\alpha)]^{\otimes_\alpha 4} \otimes_\alpha \left({}_{0}D_{x}^{\alpha} \right)[tan_\alpha(x^\alpha)] \right]$$

$$= \left({}_{0}I_{x}^{\alpha} \right) \left[\left[[tan_\alpha(x^\alpha)]^{\otimes_\alpha 6} + 1 \right]^{\otimes_\alpha (-1)} \otimes_\alpha \left[[tan_\alpha(x^\alpha)]^{\otimes_\alpha 2} + 1 \right]^{\otimes_\alpha 2} \otimes_\alpha \left({}_{0}D_{x}^{\alpha} \right)[tan_\alpha(x^\alpha)] \right]$$

$$= \left({}_{0}I_{x}^{\alpha} \right) \left[\left[[tan_\alpha(x^\alpha)]^{\otimes_\alpha 4} - [tan_\alpha(x^\alpha)]^{\otimes_\alpha 2} + 1 \right]^{\otimes_\alpha (-1)} \otimes_\alpha \left[[tan_\alpha(x^\alpha)]^{\otimes_\alpha 2} + 1 \right] \otimes_\alpha \left({}_{0}D_{x}^{\alpha} \right)[tan_\alpha(x^\alpha)] \right]$$

$$= \left({}_{0}I_{x}^{\alpha} \right) \left[\left[[tan_\alpha(x^\alpha)]^{\otimes_\alpha 2} - 1 + [tan_\alpha(x^\alpha)]^{\otimes_\alpha (-2)} \right]^{\otimes_\alpha (-1)} \otimes_\alpha \left[1 + [tan_\alpha(x^\alpha)]^{\otimes_\alpha (-2)} \right] \otimes_\alpha \left({}_{0}D_{x}^{\alpha} \right)[tan_\alpha(x^\alpha)] \right]$$

$$= \left({}_0I_x^\alpha\right)\left[\left[\left[tan_\alpha(x^\alpha) - [tan_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right]^{\otimes_\alpha 2} + 1\right]^{\otimes_\alpha(-1)} \otimes_\alpha \left({}_0D_x^\alpha\right)\left[tan_\alpha(x^\alpha) - [tan_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right]\right]$$

$$= \left({}_0I_x^\alpha\right)\left[\left[[tan_\alpha(x^\alpha) - cot_\alpha(x^\alpha)]^{\otimes_\alpha 2} + 1\right]^{\otimes_\alpha(-1)} \otimes_\alpha \left({}_0D_x^\alpha\right)[tan_\alpha(x^\alpha) - cot_\alpha(x^\alpha)]\right]$$

$$= arctan_\alpha\left(tan_\alpha(x^\alpha) - cot_\alpha(x^\alpha)\right) + \begin{cases} \frac{T_\alpha}{4} & \text{if } x > 0 \\ -\frac{T_\alpha}{4} & \text{if } x < 0 \end{cases}. \qquad \text{Q.e.d.}$$

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional calculus and a new multiplication of fractional analytic functions, we study two fractional integral problems. The exact solutions of these two fractional integrals can be obtained by using some techniques. Moreover, our results are generalizations of classical calculus results. In the future, we will continue to use Jumarie type of R-L fractional calculus and the new multiplication of fractional analytic functions to solve the problems in engineering mathematics and fractional differential equations.

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